CONCERNING CELLULAR DECOMPOSITIONS OF 3-MANIFOLDS THAT YIELD 3-MANIFOLDS

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1. **Introduction.** In this paper, we shall study cellular decompositions of 3-manifolds for which the associated decomposition space is also a 3-manifold. In [4], Bing raised the following question: Does each point-like decomposition of E^3 yield E^3 if it yields a 3-manifold? This question can be generalized as follows: If a cellular decomposition of a 3-manifold M yields a 3-manifold N, then are M and N homeomorphic? We shall establish conditions sufficient to insure that, under the hypothesis stated, M and N are homeomorphic.

The main theorem of this paper can be stated as follows. Suppose that M is a connected 3-manifold and G is a cellular decomposition of M such that the associated decomposition space is a 3-manifold N. Let P denote the projection map from M onto N, and let H_G denote the union of all the nondegenerate elements of G. Thus $\mathrm{Cl}\, P[H_G]$ is the set of singular points of the map P. Our main result then states that if N has a triangulation T such that (1) no vertex of T belongs to $\mathrm{Cl}\, P[H_G]$ and (2) for each 3-simplex σ of T, $P^{-1}[\sigma]$ lies in an open 3-cell in M, then M and N are homeomorphic.

As a corollary of the main result, we obtain the following result concerning E^3. If G is a point-like decomposition of E^3 such that $P[H_G]$ is nowhere dense and the associated decomposition space is a 3-manifold, then the decomposition space is homeomorphic to E^3. In particular, if G is a point-like decomposition of E^3 such that $P[H_G]$ lies in a closed set of dimension two, then the resulting decomposition space is homeomorphic to E^3 provided it is a 3-manifold. These results, and others given in §7, give partial solutions to Bing's question stated in the first paragraph of this section.

The results of this paper may be regarded as results concerning cellular maps from a 3-manifold onto a 3-manifold. If M is a 3-manifold, a cellular map f from M into a space is a continuous function with domain M such that if $y \in f[M]$, $f^{-1}[y]$ is a cellular subset of M. The following is a corollary of the main result: If f is a cellular map from S^3 onto a 3-manifold N and the set of all singular points of f is properly contained in N, then N is homeomorphic to S^3. Other results of this type are given in §8.

Presented to the Society, November 27, 1965; received by the editors May 20, 1965 and, in revised form, May 8, 1967.

(1) Supported in part by Air Force Office of Scientific Research Grant AF-AFOSR-359-63 and National Science Foundation Grant No. GP-3857.

Some previous results concerning point-like decompositions of E^3 and S^3 for which the decomposition space is a 3-manifold are given in [1], [2], [3], [8] and[12]. Theorem 3 of [2] and the result announced in [3] are corollaries of the theorem of this paper.

The proof of the main result is given in §6. In §§3, 4 and 5, some preliminary lemmas are established. Sections 7 and 8 give various corollaries of the main result. Those of §7 are formulated in terms of cellular decompositions while those of §8 are stated in terms of cellular maps of compact 3-manifolds.

2. **Notation and terminology.** Suppose that n is a positive integer. The statement that X is an *n-manifold* means that X is a separable metric space, each point of which has an open neighborhood homeomorphic to E^n. The statement that X is an *n-manifold-with-boundary* means that X is a separable metric space such that each point of X has a neighborhood which is an n-cell. If X is an n-manifold-with-boundary and $p \in X$, then p is a *boundary point* of X if and only if p has no open neighborhood homeomorphic to E^n. The *boundary*, denoted by Bd X, of the n-manifold-with-boundary X, is the set of all boundary points of X. The *interior*, denoted Int X, of the n-manifold-with-boundary X, is $(X-\text{Bd } X)$.

If M is either a manifold or a manifold-with-boundary, by a *triangulation* of M is meant a simplicial complex T such that (1) $M=\bigcup \{t : t \in T\}$ and (2) T is locally finite in the sense that each point of M has a neighborhood which intersects only finitely many sets of T.

Suppose that X is a topological space and G is an upper semicontinuous decomposition of X. Then X/G denotes the associated decomposition space, and P denotes the projection map from X onto X/G. The union of all the nondegenerate elements of G is denoted by H_G.

If for some positive integer n, X is either E^n or S^n, then the statement that the subcontinuum A of X is *point-like* (in X) means that if p is any point of X, then $X-A$ is homeomorphic to $X-\{p\}$.

Suppose that n is a positive integer and M is an n-manifold. The statement that the subset A of M is *cellular in M* means that there exists a sequence $C_1, C_2, C_3, \ldots$ of n-cells in M such that

(1) if i is any positive integer, $C_{i+1} \subset \text{Int } C_i$, and

(2) $A=\bigcap_{i=1}^{\infty} C_i$.

It is clear that if A is cellular in M, then A is a compact continuum. It is known (see [15], for instance) that if X is either S^3 or E^3, then a continuum A of X is point-like in X if and only if A is cellular in X.

If for some positive integer n, X is either E^n or S^n, the statement that G is a *point-like decomposition* of X means that G is an upper semicontinuous decomposition of X such that each element of G is a point-like continuum in X. If n is a positive integer and M is an n-manifold, the statement that G is a *cellular decomposition* of M means that G is an upper semicontinuous decomposition of M such that each element of G is cellular in M.

If D is a disc and α is an arc, the statement that α *spans* D means that the endpoints of α belong to Bd D and (Int $\alpha)\subset$ Int D. If A is an annulus and α is an arc, the statement that α *spans* A means that one endpoint of α belongs to one boundary component of A, the other endpoint of α belongs to the other boundary component of A, and (Int $\alpha)\subset$ Int A.

If M is a set, Cl M denotes the closure of M.

3. **Lemmas on cellular decompositions.** The lemmas of this section are of fundamental importance in the proof of the main result of this paper. Lemmas 1 and 2 are due essentially to T. M. Price [12], [13].

LEMMA 1. *If G is a cellular decomposition of a 3-manifold M and U is a simply connected open set in M/G, then $P^{-1}[U]$ is simply connected.*

Proof. The proof given for Theorem 2.1 of [13] carries over to this case.

LEMMA 2. *Suppose G is a cellular decomposition of a 3-manifold M and W is an open 3-cell in M/G. If $P^{-1}[W]$ lies in an open 3-cell in M, then $P^{-1}[W]$ is an open 3-cell.*

Proof. The argument for the case $n=3$ of Theorem 2.2 of [13] carries over with no change. Since $P^{-1}[W]$ lies in an open 3-cell, it follows that $P^{-1}[W]$ is an open 3-cell.

LEMMA 3. *Suppose that G is a cellular decomposition of a 3-manifold M such that M/G is a 3-manifold N. Suppose that T is a triangulation of N, σ is a 3-simplex of T, and $P^{-1}[\sigma]$ lies in an open 3-cell U in M. Then if V is any neighborhood of σ in N, there is a polyhedral 3-cell C in N such that $\sigma\subset$ Int C, $C\subset V$, $P^{-1}[C]\subset U$, and P^{-1}[Int C] is an open 3-cell.*

Proof. There is a polyhedral 3-cell C in N such that C is a regular neighborhood of σ in N, $\sigma\subset$ Int C, $C\subset V$, and $P^{-1}[C]\subset U$. By Lemma 2, P^{-1}[Int C] is an open 3-cell.

4. **Lemmas on polyhedral 2-spheres.**

LEMMA 4. *Suppose that X is a polyhedral 2-sphere in a triangulated 3-manifold M such that X separates M. Suppose that x and y are distinct points of X, and γ is a polygonal simple closed curve such that x and y belong to γ, and distinct components of $\gamma-\{x, y\}$ lie in different components of $M-X$. Suppose that Δ is a polyhedral singular disc such that $\gamma=$ Bd Δ, and Δ and X are in relative general position. Then there is an arc α from x to y and contained in $\Delta\cap X$.*

Proof. Let Δ_0 be a 2-simplex and let f be a continuous function from Δ_0 onto Δ such that $f|$Bd Δ_0 is a homeomorphism from Bd Δ_0 onto γ. Let x_0 and y_0 be points of Bd Δ_0 such that $f(x_0)=x$ and $f(y_0)=y$. Let p and q be points of different components of $\gamma-X$, and let p_0 and q_0 be points of Bd Δ_0 such that $f(p_0)=p$ and $f(q_0)=q$. Note that $\{x_0, y_0\}$ separates p_0 from q_0 on Bd Δ_0.

There is a component A of $f^{-1}[\Delta \cap X]$ such that A separates p_0 and q_0 in Δ_0. For suppose not. Since $f^{-1}[\Delta \cap X]$ has only finitely many components, then, if no one of them separates p_0 and q_0 in Δ_0, then $f^{-1}[\Delta \cap X]$ does not separate p_0 and q_0 in Δ_0 [10, Chapter IV, Theorem 19]. There is, in that case, an arc β in Δ_0 from p_0 to q_0 and lying in $\Delta_0 - f^{-1}[\Delta \cap X]$. Then $f[\beta]$ is a continuum in Δ joining p and q and disjoint from X. Since p and q belong to different components of $M-X$, $f[\beta]$ intersects X. This is a contradiction, so there is a component A of $f^{-1}[\Delta \cap X]$ such that A separates p_0 and q_0 on Δ_0. Further, both x_0 and y_0 belong to A since clearly A contains at least two distinct points of $\operatorname{Bd} \Delta_0$ and $\{x_0, y_0\} = (\operatorname{Bd} \Delta_0) \cap f^{-1}[\Delta \cap X]$.

Hence $f[A]$ is a connected finite graph contained in $\Delta \cap X$ and containing x and y. Then $f[A]$ contains an arc α from x to y satisfying the conclusion of Lemma 4.

Suppose that C is a polyhedral 3-cell in a triangulated 3-manifold M, and Δ is a polyhedral singular disc in M. The statement that Δ is in *normal position relative to* C means that (1) $(\operatorname{Bd} \Delta) \cap (\operatorname{Bd} C)$ is a polygonal arc α and (2) Δ is the union of a polyhedral disc E and a polyhedral singular disc F such that (a) $\alpha \subset \operatorname{Bd} E$, (b) $E \cap C = \alpha$, (c) $\operatorname{Bd} F$ misses C, (d) $(\operatorname{Bd} E) \cap (\operatorname{Bd} F)$ is an arc β such that β misses α, and (e) F and $\operatorname{Bd} C$ are in relative general position.

LEMMA 5. *Suppose that C is a polyhedral 3-cell in a triangulated 3-manifold M, and A is a polyhedral annulus on* Bd C. *Suppose that $n > 1$ and $\Delta_1, \Delta_2, \ldots,$ and Δ_n are mutually disjoint polyhedral singular discs in M such that if $i = 1, 2, \ldots,$ or n, (1) Δ_i is in normal position relative to C, (2) $(\operatorname{Bd} C) \cap (\operatorname{Bd} \Delta_i)$ is an arc α_i spanning A, and (3) $\Delta_i \cap (\operatorname{Bd} C) \subset (\operatorname{Int} A) \cup (\operatorname{Bd} \alpha_i)$. Then if U is any open neighborhood, in M, of* Int A, *there exist mutually disjoint polyhedral singular discs $\Delta'_1, \Delta'_2, \ldots,$ and Δ'_n such that if $i = 1, 2, \ldots,$ or n, (1) $\operatorname{Bd} \Delta'_i = \operatorname{Bd} \Delta_i$, (2) $\Delta'_i \cap C = \alpha_i$, (3) $\Delta'_i \subset \Delta_i \cup U$, and Δ'_i is in normal position relative to C.*

Proof. If $i = 1, 2, \ldots,$ or n, let E_i be a polyhedral disc and F_i be a polyhedral singular disc such that $\Delta_i = E_i \cup F_i$, $\alpha_i \subset \operatorname{Bd} E_i$, and E_i and F_i satisfy conditions relative to C and one another required in the definition of normal position. In particular, Bd F_i misses C.

Let Δ_0 be a 2-simplex and if $i = 1, 2, \ldots,$ or n, let f_i be a continuous function from Δ_0 onto F_i such that (1) $f_i | \operatorname{Bd} \Delta_0$ is a homeomorphism from $\operatorname{Bd} \Delta_0$ onto Bd F_i and (2) f is locally a piecewise-linear homeomorphism. If $i = 1, 2, \ldots,$ or n, each component of $f_i^{-1}[(\operatorname{Bd} C) \cap F_i]$ is a simple closed curve lying in Int Δ_0. If μ is such a simple closed curve, $f_i[\mu]$ will be called a *curve of intersection* of F_i with Bd C.

Since $\Delta_1, \Delta_2, \ldots,$ and Δ_n are mutually disjoint, it follows that if $i = 1, 2, \ldots,$ or n, $F_i \cap (\operatorname{Bd} C)$ lies in $(\operatorname{Int} A) - (\alpha_1 \cup \cdots \cup \alpha_{i-1} \cup \alpha_{i+1} \cup \cdots \cup \alpha_n)$. Further, each component of $(\operatorname{Int} A) - (\alpha_1 \cup \cdots \cup \alpha_{i-1} \cup \alpha_{i+1} \cup \cdots \cup \alpha_n)$ is an open disc.

Suppose that $i = 1, 2, \ldots,$ or n, and γ is a curve of intersection of F_i with Bd C. With the aid of [17, Chapter VI, Theorem 3.11], it may be shown that there exists a simple closed curve J_γ on Bd C such that (1) J_γ lies in some component D_γ of

(Int A)$-(\alpha_1 \cup \cdots \cup \alpha_{i-1} \cup \alpha_{i+1} \cup \cdots \cup \alpha_n)$, (2) the component C_γ of $F_i \cap \mathrm{Bd}\, C$ containing γ lies in the disc B_γ in D_γ bounded by J_γ, and (3) the component of $B_\gamma - C_\gamma$ containing J_γ is disjoint from $\bigcup_{k=1}^{n} F_k$.

Since each of $F_1, F_2, \ldots$, and F_n has only finitely many curves of intersection with Bd C, there is a positive integer j such that $j \leqq n$ and a curve of intersection λ of F_j with Bd C such that if $k \neq j$, B_λ is disjoint from Δ_k. Now λ bounds a singular disc δ_λ in Int B_λ. We cut F_j along λ and then replace the singular disc on F_j bounded by λ by δ_λ. We then deform the resulting singular disc slightly to one side of Bd C, staying in U during the deformation. This is done in such a way that there results a polyhedral singular disc $F_j^{(1)}$ such that (1) Bd $F_j^{(1)} = \mathrm{Bd}\, F_j$ and $F_j^{(1)} \subset F_j \cup U$, (2) λ is not a curve of intersection of $F_j^{(1)}$ with Bd C, and (3) if γ is a curve of intersection of $F_j^{(1)}$ with Bd C, then γ is a curve of intersection of F_j with Bd C. Observe that if $k \neq j$, $F_j^{(1)}$ and Δ_k are disjoint.

The singular discs $F_1, \ldots, F_{j-1}, F_j^{(1)}, F_{j+1}, \ldots$, and F_n satisfy the conditions assumed for $F_1, F_2, \ldots$, and F_n. Hence the process described above may be repeated. After finitely many steps, there result mutually disjoint polyhedral singular discs $F_1', F_2', \ldots$, and F_n' such that if $i = 1, 2, \ldots$, or n, (1) Bd $F_1' = \mathrm{Bd}\, F_i$, (2) $F_i' \subset F_i \cup U$, and (3) F_i' misses C. If $i = 1, 2, \ldots$, or n, adjust F_i' slightly so that if Δ_i' is the union of E_i and the adjusted F_i', Δ_i' is a singular disc in normal position relative to C. Then $\Delta_1', \Delta_2', \ldots$, and Δ_n' satisfy the conclusion of Lemma 5.

5. **Constructing homeomorphisms.** In order to prove the theorem, we shall need to construct a homeomorphism from one 3-manifold M onto another 3-manifold N. In this section, we shall show that such a homeomorphism can be constructed provided we can embed into M the carrier of the 2-skeleton of some triangulation of N so that certain conditions are satisfied. These conditions are stated in Lemma 8. Lemmas 6 and 7 are preliminary lemmas for Lemma 8.

Once Lemma 8 is established, the proof of the theorem is reduced to the construction of a suitable embedding into M of the carrier of the 2-skeleton of some triangulation of N.

LEMMA 6. *Suppose that M is a* 3*-manifold and T is a triangulation of M. If T_2 is the carrier of the* 2*-skeleton of T and σ is any* 3*-simplex of T, then $T_2 - (\mathrm{Bd}\, \sigma)$ is connected.*

Proof. Suppose that v and v' are any two distinct vertices of T not on σ. It will be shown that $T_2 - \mathrm{Bd}\, \sigma$ contains a connected set containing both v and v'.

Since M is connected, there is a chain $\{s_1, s_2, \ldots, s_m\}$ of 1-simplexes of T such that

(1) v is a vertex of s_1 and v' is a vertex of s_m,

(2) $s_1 \cup s_2 \cup \cdots \cup s_m$ is an arc A, and

(3) $A \cap (\mathrm{Bd}\, \sigma)$ is either (a) a vertex p of T or (b) a 1-simplex s_i of T.

Suppose 3(a) holds and consider the closed star St (p) of p in T. By Theorem 1 of [9], Bd St (p) is a 2-sphere S. Clearly (Bd σ) $\cap\, S$ is a 2-simplex D. Now $A \cap S$ is disjoint from D and it is clear that both components of $A-\{p\}$ intersect S. Therefore $(A-\{p\}) \cup (S-D)$ is a connected subset of $T_2-(\text{Bd } \sigma)$ containing both v and v'.

Suppose 3(b) holds, and that $s_i = \langle p_1 p_2 \rangle$ where p_1 precedes p_2 on A in the order from v to v' on A. Let A_1 and A_2 be the components of $A-\langle p_1 p_2 \rangle$ containing v and v', respectively. Let S_1 and S_2 denote Bd St (p_1) and Bd St (p_2), respectively; S_1 and S_2 are 2-spheres.

There is a 3-simplex σ' of T distinct from σ and having both p_1 and p_2 as vertices. Let q be a vertex of σ' not on σ; q is distinct from both p_1 and p_2. Clearly $q \in S_1 \cap S_2$. If $j=1$ or 2, $(\text{Bd } \sigma) \cap S_i$ is a disc D_i and $q \notin D_i$. Further, if $j=1$ or 2, A_j intersects $S_j - D_j$. Hence

$$A_1 \cup A_2 \cup (S_1 - D_1) \cup (S_2 - D_2)$$

is a connected subset of $T_2-(\text{Bd } \sigma)$ containing both v and v'.

Hence for each two vertices v and v' of T not on σ, there is a connected subset of $T_2-(\text{Bd } \sigma)$ containing both v and v'. Since each component of $T_2-(\text{Bd } \sigma)$ contains at least one vertex of T, it follows that $T_2-(\text{Bd } \sigma)$ is connected.

Lemma 7. *If M and N are 3-manifolds, Σ and T are triangulations of M and N, respectively, and Σ and T are isomorphic (as complexes), then M and N are homeomorphic.*

Proof. Let φ be an isomorphism from Σ onto T. Define a function f as follows: If v is a vertex of Σ, then $f(v)=\varphi(v)$. Extend f homeomorphically to the 1-simplexes of Σ, then to the 2-simplexes of Σ, and finally to the 3-simplexes of Σ so that if σ is any simplex of Σ, $f[\sigma]=\varphi(\sigma)$. It is clear that the function f so constructed is from M onto N and is one-to-one. Since Σ is locally finite, f is continuous, and since T is locally finite, f^{-1} is continuous. Hence f is a homeomorphism from M onto N.

Lemma 8. *Suppose that M and N are triangulated connected 3-manifolds, T is a triangulation of N, and T_2 is the carrier of the 2-skeleton of T. Suppose there exist a piece-wise linear homeomorphism h from T_2 into M and a locally finite collection $\mathcal{U}$ of open 3-cells in M such that* (1) *if σ is any 3-simplex of T, there is an open 3-cell U_σ of $\mathcal{U}$ such that $h[\text{Bd } \sigma] \subset U_\sigma$, but $h[T_2]$ does not lie in U_σ, and* (2) *if σ and τ are distinct 3-simplexes of T, $U_\sigma \neq U_\tau$. Then M and N are homeomorphic.*

Proof. We shall construct a triangulation Σ of M such that Σ and T are isomorphic. Lemma 8 will then follow from Lemma 7.

For each 3-simplex σ of T, let S_σ denote $h[\text{Bd } \sigma]$. By hypothesis, there is an open 3-cell U_σ of $\mathcal{U}$ such that $S_\sigma \subset U$. Let I_σ be the component of $U_\sigma - S_\sigma$ whose closure relative to U_σ is compact. Then $S_\sigma \cup I_\sigma$ is a polyhedral 3-cell in M. It is easy to see that S_σ separates I_σ and $M-(S_\sigma \cup I_\sigma)$, and that $M-(S_\sigma \cup I_\sigma)$ is connected.

We shall show now that I_σ and $h[T_2]$ are disjoint. By Lemma 6, $T_2 - \text{Bd } \sigma$ is connected. Hence $h[T_2]-S_\sigma$ is connected. If T_2 intersects I_σ, then $h[T_2]-S_\sigma$ lies in I_σ and hence in U_σ. It would follow then that $h[T_2] \subset U_\sigma$, but this is contrary to hypothesis.

Next we shall show that if σ and τ are distinct 3-simplexes of T, I_σ and I_τ are disjoint. Suppose I_σ and I_τ intersect. Since I_σ and I_τ are connected and, as shown above, neither intersects the boundary of the other, one lies in the other. Suppose that $I_\sigma \subset I_\tau$. Since $S_\sigma \neq S_\tau$, then $I_\sigma \neq I_\tau$. Hence some boundary point of I_σ lies in I_τ. This is contradictory since $h[T_2]$ and I_τ are disjoint. Similarly, it is impossible that $I_\tau \subset I_\sigma$. Hence I_σ and I_τ are disjoint.

Let Σ denote $\{h[t] : t$ *is a* 0-, 1-, *or* 2-*simplex of* $T\} \cup \{S_\sigma \cup I_\sigma : \sigma$ *is a* 3-*simplex of* $T\}$. It is clear that Σ and T are isomorphic as complexes. We shall show that Σ is a triangulation of M.

First, Σ is a locally finite collection. This follows from the following facts: (1) $\mathscr{U}$ is locally finite. (2) If σ and τ are distinct 3-simplexes of T, $U_\sigma \neq U_\tau$. (3) For each 3-simplex σ of T, $S_\sigma \cup I_\sigma \subset U_\sigma$.

Now let Σ_3 denote $\bigcup \{S_\sigma \cup I_\sigma : \sigma$ *is a* 3-*simplex of* $T\}$. It will be shown that $\Sigma_3 = M$. Suppose that Σ_3 is a proper subset of M. By Lemma 7, there is a homeomorphism g from N onto Σ_3. Since Σ is a locally finite collection, Σ_3 is closed in M. Since M is connected, there is a point x of Σ_3 which is a limit point of $M - \Sigma_3$. There is an open 3-cell V in N such that V is an open neighborhood in N of $g^{-1}(x)$. Since M is a 3-manifold, $g[V]$ is open in M and hence intersects $M - \Sigma_3$. But $g[V] \subset \Sigma_3$, and this is a contradiction. Therefore $\Sigma_3 = M$.

It then follows that Σ is a triangulation of M. Since Σ and T are isomorphic, then by Lemma 7, M and N are homeomorphic.

6. **Proof of the main result.**

THEOREM. *Suppose that M is a connected 3-manifold, G is a cellular decomposition of M such that M/G is a 3-manifold N, and N has a triangulation T such that* (1) *no vertex of T belongs to the closure of $P[H_G]$ and* (2) *if σ is any 3-simplex of T, there is an open 3-cell in M containing $P^{-1}[\sigma]$. Then M and N are homeomorphic.*

The proof of the theorem is based on Lemma 8. M has a triangulation [5], [9]. We shall construct (1) a piece-wise linear embedding h of the carrier T_2 of the 2-skeleton of T into M and (2) a locally finite collection $\mathscr{U}$ of open 3-cells in M such that h and $\mathscr{U}$ satisfy the hypothesis of Lemma 8. It will then follow that M and N are homeomorphic.

In order to construct the embedding h of T_2 into M, we construct a polyhedral 2-complex Σ in M such that Σ and the 2-skeleton of T are isomorphic complexes and Σ satisfies certain additional technical conditions.

Speaking roughly, we shall construct the 2-complex Σ by using P^{-1} to "lift" T_2 into M. It should be emphasized, however, that P^{-1} is by no means necessarily a local homeomorphism. Hence our procedure will be, roughly, to use $P^{-1}[T_2]$ as a guide in constructing Σ. For each simplex s of T of dimension 0, 1, or 2, we approximate $P^{-1}[s]$ with a polyhedral set. These approximations may have singularities. From these polyhedral sets we will then construct the 2-complex Σ.

The major difficulty is in constructing the 2-simplexes of Σ and insuring that they fit together properly. As an aid toward the construction of Σ, we divide T_2 into two pieces and construct the 2-simplexes of Σ in two corresponding pieces. We construct a tubular neighborhood K of the carrier T_1 of the 1-skeleton of T. We then use P^{-1} to "lift" K into M to yield a tubular neighborhood K' of the carrier Σ_1 of the 1-skeleton of Σ. We construct each 2-simplex of Σ in two steps: the part of the 2-simplex in K', and the part not in K'.

Therefore the proof can be divided into three main parts. The first is the construction of the tubular neighborhood K of T_1 and certain associated sets. The second is the construction of the tubular neighborhood K' of Σ_1 in M. The last is the construction of the simplexes of Σ and of the embedding h of T_2 into M.

Proof of the theorem. N has, by hypothesis, a triangulation T, and the statement that a subset of N is polygonal or polyhedral means with respect to T. Later in the proof we shall choose a triangulation of M.

In the proof of the theorem, we use a process of thickening arcs, discs, and other subsets of N. This process, not described in detail in this paper, may be carried out with the use of Whitehead's theory of regular neighborhoods and suitable subdivisions of the triangulation T.

The 2-complex to be constructed in M is denoted by Σ. If $i=0$, 1, or 2, T_i and Σ_i denote the carriers of the i-skeletons of T and Σ, respectively.

Since the proof of the theorem is long, we have broken the proof into 12 steps.

Step 1. For each 3-simplex σ of T, we shall construct an open 3-cell U_σ in M having certain properties. First we choose certain 3-cells in N. If σ is a 3-simplex of T, there is, by hypothesis, an open 3-cell U_σ^0 in M such that $P^{-1}[\sigma]\subset U_\sigma^0$. Since G is upper semicontinuous, there is an open set V_σ^0 in N containing σ and such that $P^{-1}[V_\sigma^0]\subset U_\sigma^0$. Let $\mathscr{W}$ be the set of all 3-simplexes of T; $\mathscr{W}$ is a countable locally finite collection of closed sets covering N. It may be shown that there is a locally finite collection $\mathscr{A}$ of open sets such that if $\sigma\in\mathscr{W}$, there is a set of $\mathscr{A}$ containing σ and contained in V_σ^0. For each 3-simplex σ of T, let W_σ^0 be an open set of $\mathscr{A}$ such that $\sigma\subset W_\sigma^0$ and $W_\sigma^0\subset V_\sigma^0$.

If σ is a 3-simplex of T, there is, by Lemma 3, a 3-cell X_σ in N such that $\sigma\subset \operatorname{Int} X_\sigma$, $P^{-1}[\operatorname{Int} X_\sigma]\subset U_\sigma^0$, $P^{-1}[\operatorname{Int} X_\sigma]$ is an open 3-cell, and X_σ lies in the open star (in T) of σ. For each 3-simplex σ of T, let U_σ denote $P^{-1}[\operatorname{Int} X_\sigma]$. Then

$$\{U_\sigma : \sigma \textit{ is a } 3\textit{-simplex of } T\}$$

is a locally finite collection of open 3-cells covering M such that if σ is any 3-simplex of T, $P^{-1}[\sigma]\subset U_\sigma$. Note that for each 3-simplex σ of T, X_σ contains no vertex of T distinct from the vertices of σ.

Step 2. We shall construct a tubular neighborhood K of T_1. Let $v_1, v_2, \ldots$ denote the distinct vertices of T. Let $C_1, C_2, \ldots$ be mutually disjoint polyhedral 3-cells in N such that if v_i is any vertex of T, then

(1) $v_i\in \operatorname{Int} C_i$ and C_i is disjoint from the closure of $P[H_G]$,

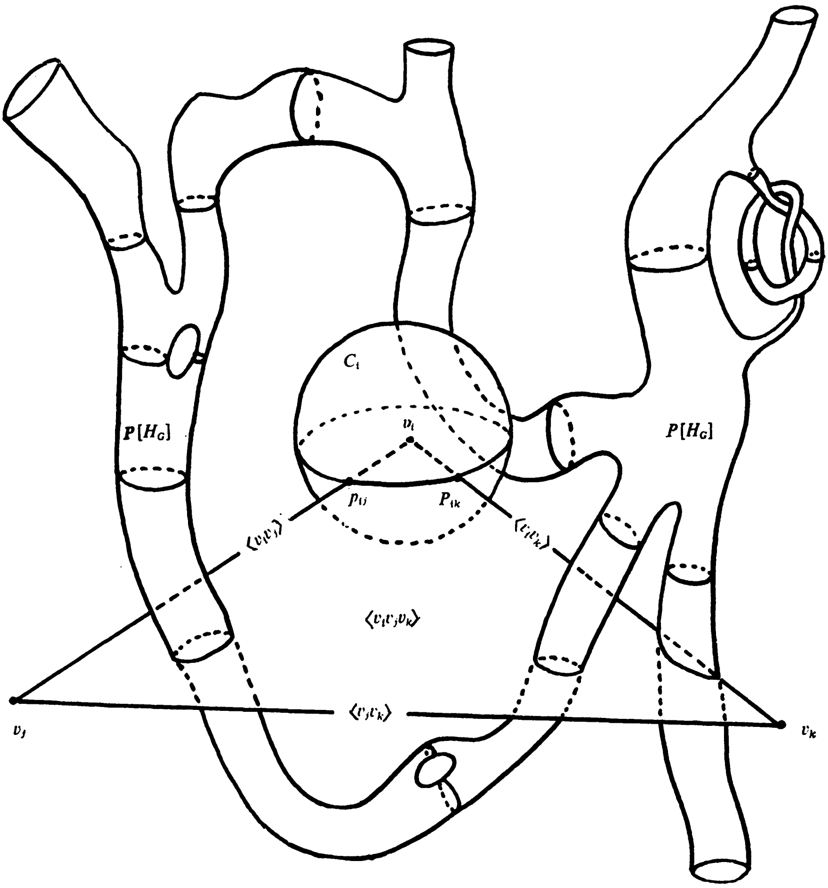

FIGURE 1

(2) If v_j is a vertex of T such that $\langle v_iv_j\rangle$ is a 1-simplex of T, there is a point p_{ij} of Int $\langle v_iv_j\rangle$ such that $\langle v_iv_j\rangle \cap \text{Bd } C=\{p_{ij}\}$,

(3) C_i intersects various 1- and 2-simplexes of T as indicated in Figure 1, and

(4) if σ is any 3-simplex of T having v_i as a vertex, then $C_i \subset \text{Int } X_\sigma$, but if σ is any 3-simplex of T not having v_i as a vertex, then C_i and X_σ are disjoint.

Let $\langle v_{i_1}v_{j_1}\rangle, \langle v_{i_2}v_{j_2}\rangle, \ldots$ denote the distinct 1-simplexes of T. Let $L_{i_1j_1}, L_{i_2j_2}, \ldots$ be mutually disjoint polyhedral 3-cells in N such that if $\langle v_iv_j\rangle$ is a 1-simplex of T,

(1) $L_{ij}=L_{ji}$,

(2) $C_i \cap L_{ij}$ is a polyhedral disc Δ_{ij} such that $\Delta_{ij}=\text{Bd } C_i \cap \text{Bd } L_{ij}$ and $p_{ij} \in \text{Int } \Delta_{ij}$,

(3) if $t=1, 2, \ldots$ and t is neither i nor j, then L_{ij} and C_t are disjoint,

(4) L_{ij} intersects $\langle v_iv_j\rangle$ and various 2-simplexes of T as indicated in Figures 2 and 3, and

(5) if σ is any 3-simplex of T having $\langle v_iv_j\rangle$ as an edge, then $L_{ij} \subset \text{Int } X_\sigma$.

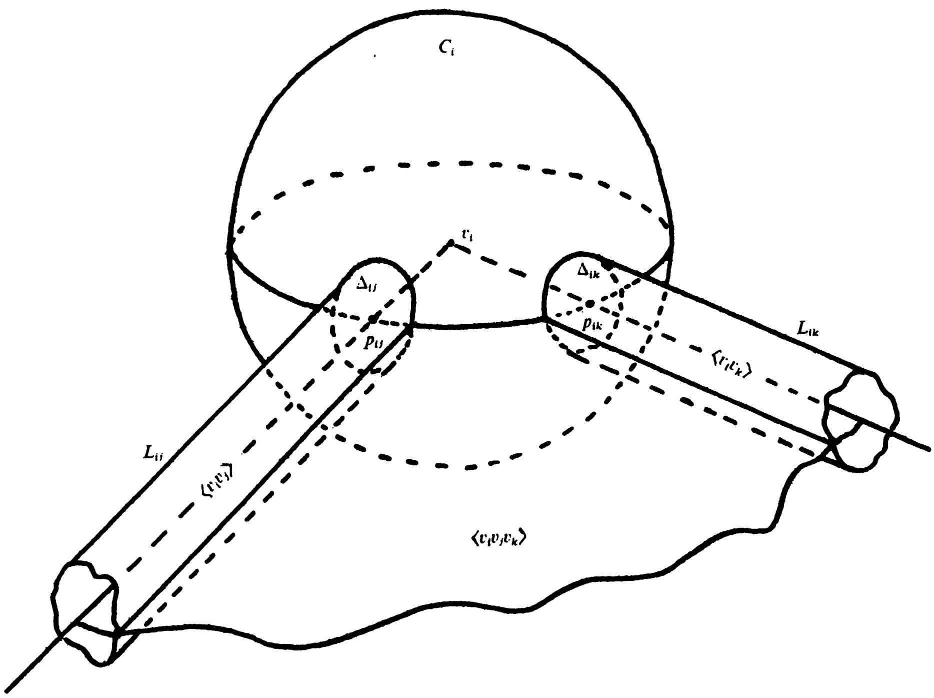

FIGURE 2

The tubular neighborhood K of T_1 is

$$(\bigcup \{C_i : v_i \textit{ is a vertex of } T) \cup (\bigcup \{L_{ij} : \langle v_iv_j\rangle \textit{ is a 1-simplex of } T\}).$$

Step 3. Suppose $\langle v_iv_j\rangle$ is a 1-simplex of T. We shall describe some subsets of $\text{Bd}\, L_{ij}$. Let A_{ij} denote

$$\text{Bd}\, L_{ij} - (\text{Int}\, \Delta_{ij} \cup \text{Int}\, \Delta_{ji});$$

A_{ij} is an annulus whose boundary curves are $\text{Bd}\, \Delta_{ij}$ and $\text{Bd}\, \Delta_{ji}$. Note that $A_{ij} = A_{ji}$. If $\langle v_iv_jv_k\rangle$ is any 2-simplex having $\langle v_iv_j\rangle$ as an edge, let π_{ijk} denote the arc $A_{ij} \cap \langle v_iv_jv_k\rangle$; $\pi_{ijk} = \pi_{jik}$ and π_{ijk} spans A_{ij}. See Figure 4.

Recall that C_i and C_j are disjoint from $\text{Cl}\, P[H_G]$. There exist, then, disjoint annuli B_{ij} and B_{ji} on A_{ij} such that

(1) B_{ij} and B_{ji} are disjoint from $\text{Cl}\, P[H_G]$,

(2) $\text{Bd}\, \Delta_{ij}$ is one boundary component of B_{ij}, and $\text{Bd}\, \Delta_{ji}$ is one boundary component of B_{ji}, and

(3) B_{ij} and B_{ji} intersect the various arcs π_{ijk} as indicated in Figure 4.

Let F_{ij} denote $\text{Cl}\, [A_{ij} - (B_{ij} \cup B_{ji})]$; F_{ij} is an annulus contained in $\text{Int}\, A_{ij}$.

Let $\langle v_iv_jv_{k_1}\rangle, \langle v_iv_jv_{k_2}\rangle, \ldots,$ and $\langle v_iv_jv_{k_{a(ij)}}\rangle$ denote the 2-simplexes of T having $\langle v_iv_j\rangle$ as an edge. There exist disjoint polyhedral discs $Q_{ijk_1}, Q_{ijk_2}, \ldots,$ and $Q_{ijk_{a(ij)}}$ such that if $k = k_1, k_2, \ldots,$ or $k_{a(ij)}$,

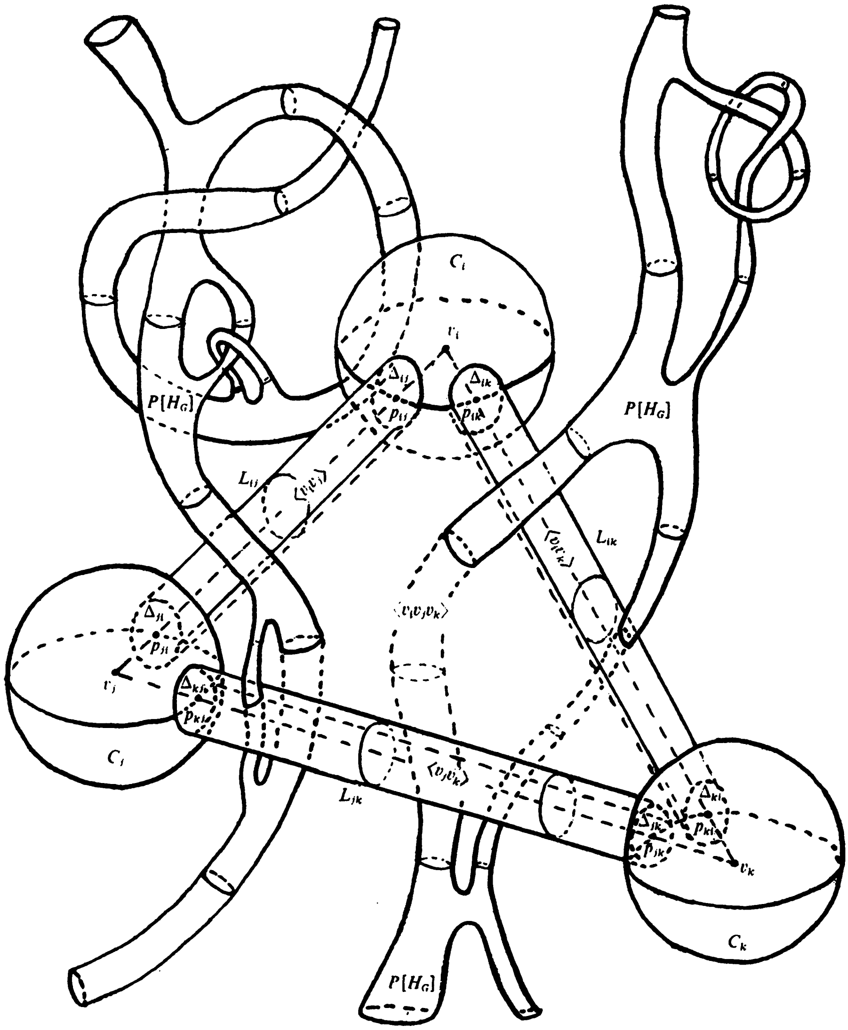

FIGURE 3

(1) $Q_{ijk} \subset A_{ij}$ and Bd Q_{ijk} is the union of two spanning arcs of A_{ij}, an arc on Bd Δ_{ij}, and an arc on Bd Δ_{ji},

(2) the arc π_{ijk} spans Q_{ijk}, and

(3) Q_{ijk} lies on A_{ij} as shown in Figure 5.

Let $\mathscr{E}_{ij}$ denote the set of all E such that either

(1) for some positive integer k such that $\langle v_i v_j v_k \rangle$ is a 2-simplex of T, $E = F_{ij} \cap Q_{ijk}$, or

(2) E is the closure of a component of

$$F_{ij} - \bigcup \{Q_{ijk} : \langle v_i v_j v_k \rangle \textit{ is a 2-simplex of } T\}.$$

Each set of $\mathscr{E}_{ij}$ is a disc lying on F_{ij} and such that $E \cap$ Bd F_{ij} is the union of two

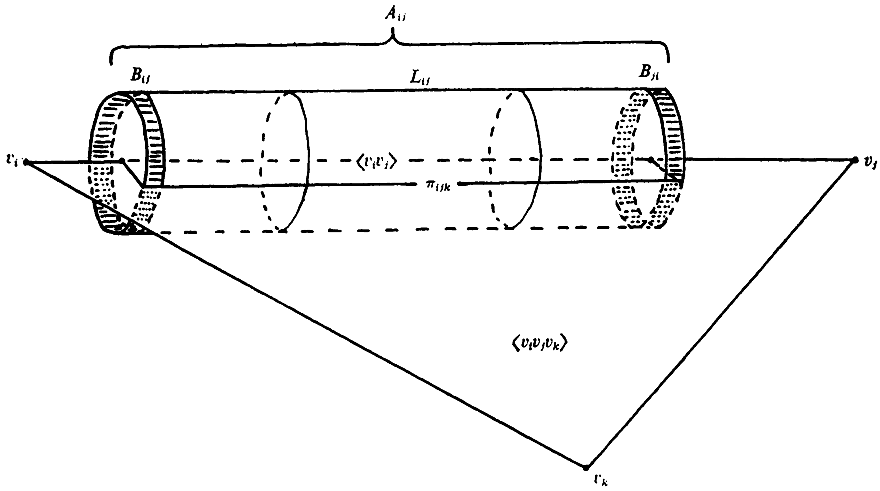

FIGURE 4

disjoint arcs, one on each boundary component of F_{ij}. Further, there is a cyclic ordering

$$E_{ij1}, E_{ij2}, \ldots, E_{ij\lambda_{ij}}$$

of the sets of $\mathscr{E}_{ij}$ such that if each of k and l is a positive integer not greater than λ_{ij}, then

(1) E_{ijk} and E_{ijl} intersect if and only if either $|k-l|\leq 1$ or $\{k, l\}=\{1, \lambda_{ij}\}$, and

(2) if E_{ijk} and E_{ijl} intersect, then $E_{ijk}\cap E_{ijl}$ is an arc spanning the annulus F_{ij}.

Step 4. We shall now describe certain open subsets of N associated with Bd K. Consider the annuli $F_{i_1j_1}, F_{i_2j_2}, \ldots$. There exist mutually disjoint open sets (in N) $F^*_{i_1j_1}, F^*_{i_2j_2}, \ldots$ such that if $t=1, 2, \ldots$,

(1) $F_{i_tj_t}\subset F^*_{i_tj_t}$,

(2) if v_k is a vertex of T, $F^*_{i_tj_t}$ and C_k are disjoint,

(3) if $s\neq t$, $L_{i_sj_s}$ and $F^*_{i_tj_t}$ are disjoint,

(4) $F^*_{i_tj_t}$ is disjoint from T_1, and

(5) $F^*_{i_tj_t}$ is disjoint from each 2-simplex of T not having $\langle v_{i_t}v_{j_t}\rangle$ as an edge.

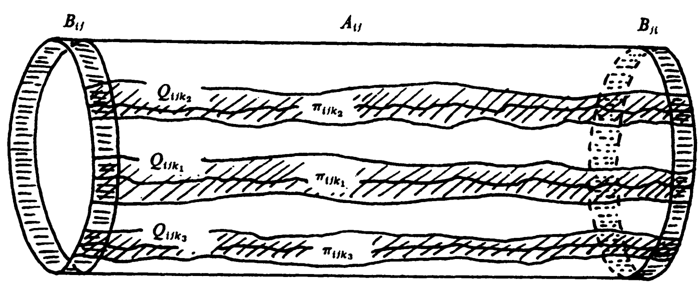

FIGURE 5

Suppose now that $\langle v_i v_j \rangle$ is a 1-simplex of T. There exist connected open sets $W_{ij1}, W_{ij2}, \ldots,$ and $W_{ij\lambda_{ij}}$ such that

(1) if $t=1, 2, \ldots,$ or λ_{ij},

(i) W_{ijt} is obtained by a slight thickening of E_{ijt},

(ii) $E_{ijt} \subset W_{ijt}$, and $W_{ijt} \subset F_{ij}^*$,

(iii) W_{ijt} is simply connected, and

(iv) if σ is any 3-simplex of T having $\langle v_i v_j \rangle$ as an edge, then $W_{ijt} \subset \text{Int } X_\sigma$, and

(2) if $s=1, 2, \ldots,$ or λ_{ij} and $t=1, 2, \ldots,$ or λ_{ij}, then W_{ijs} and W_{ijt} intersect if and only if E_{ijs} and E_{ijt} intersect, and if W_{ijs} and W_{ijt} intersect, their common part is connected.

Let W_{ij} denote $\bigcup_{t=1}^{\lambda_{ij}} W_{ijt}$. Then W_{ij} satisfies the following conditions:

(1) $F_{ij} \subset W_{ij}$.

(2) W_{ij} intersects no 2-simplex of T not having $\langle v_i v_j \rangle$ as an edge.

(3) For each vertex v_k of T, C_k and W_{ij} are disjoint.

(4) If $\langle v_s v_t \rangle$ is a 1-simplex of T distinct from $\langle v_i v_j \rangle$, L_{st} and W_{ij} are disjoint.

(5) If σ is any 3-simplex of T having $\langle v_i v_j \rangle$ as an edge, then $W_{ij} \subset \text{Int } X_\sigma$.

(6) There is a polyhedral 3-cell Y_{ij} contained in $L_{ij} - W_{ij}$ and such that

(i) $Y_{ij} \cap \Delta_{ij}$ and $Y_{ij} \cap \Delta_{ji}$ are discs such that p_{ij} and p_{ji} belong to their respective interiors, and

(ii) Y_{ij} meets each 2-simplex of T having $\langle v_i v_j \rangle$ as an edge as indicated in Figure 6.

Before we start on Step 5, we introduce some notation to be used throughout the remainder of the proof. If X is any subset of N, then $\tilde{X}$ denotes $P^{-1}[X]$. If p is any point of N and there is only one point x of M such that $P(x)=p$, then $\tilde{p}$ denotes x.

Step 5. We first choose a triangulation of M. We may, by [5], [9], regard M as a triangulated manifold. It is easy to see that for each vertex v_i, Bd $\tilde{C}_i$ is locally tame [7], and for each 1-simplex $\langle v_i v_j \rangle$, $\tilde{B}_{ij}$ and $\tilde{B}_{ji}$ are locally tame. By [7, Theorem 8], there is a triangulation of M relative to which the $\tilde{C}_i$'s, the $\tilde{B}_{ij}$'s, and various associated sets, are polyhedral. Choose one such triangulation of M; the statement that a subset of M is polygonal or polyhedral means relative to this particular triangulation of M.

Now for each 1-simplex $\langle v_i v_j \rangle$ of T, we shall construct an annulus A'_{ij} in M that corresponds to the annulus A_{ij} in N.

Suppose that $\langle v_i v_j \rangle$ is a 1-simplex of T. Let $x_1, x_2, \ldots,$ and $x_{\lambda_{ij}}$ be points of $B_{ij} \cap F_{ij}$, and let $y_1, y_2, \ldots,$ and $y_{\lambda_{ij}}$ be points of $B_{ji} \cap F_{ij}$ such that if $t=1, 2, \ldots,$ or $\lambda_{ij}-1$, the arc $E_{ijt} \cap E_{ij(t+1)}$ has endpoints x_t and y_t, and the arc $E_{ij1} \cap E_{ij\lambda_{ij}}$ has endpoints $x_{\lambda_{ij}}$ and $y_{\lambda_{ij}}$.

Consider the sets $\tilde{W}_{ij1}, \tilde{W}_{ij2}, \ldots,$ and $\tilde{W}_{ij\lambda_{ij}}$. If $t=1, 2, \ldots,$ or λ_{ij}, $W_{ijt} \cap W_{ij(t+1)}$ is connected and hence so is $\tilde{W}_{ijt} \cap \tilde{W}_{ij(t+1)}$. There is, therefore, a polygonal arc μ^0_{ijt} from $\tilde{x}_t$ to $\tilde{y}_t$, lying in $\tilde{W}_{ijt} \cap \tilde{W}_{ij(t+1)}$. Similarly, there is a polygonal arc $\mu^0_{ij\lambda_{ij}}$ from $\tilde{x}_{\lambda_{ij}}$ to $\tilde{y}_{\lambda_{ij}}$ lying in $\tilde{W}_{ij1} \cap \tilde{W}_{ij\lambda_{ij}}$.

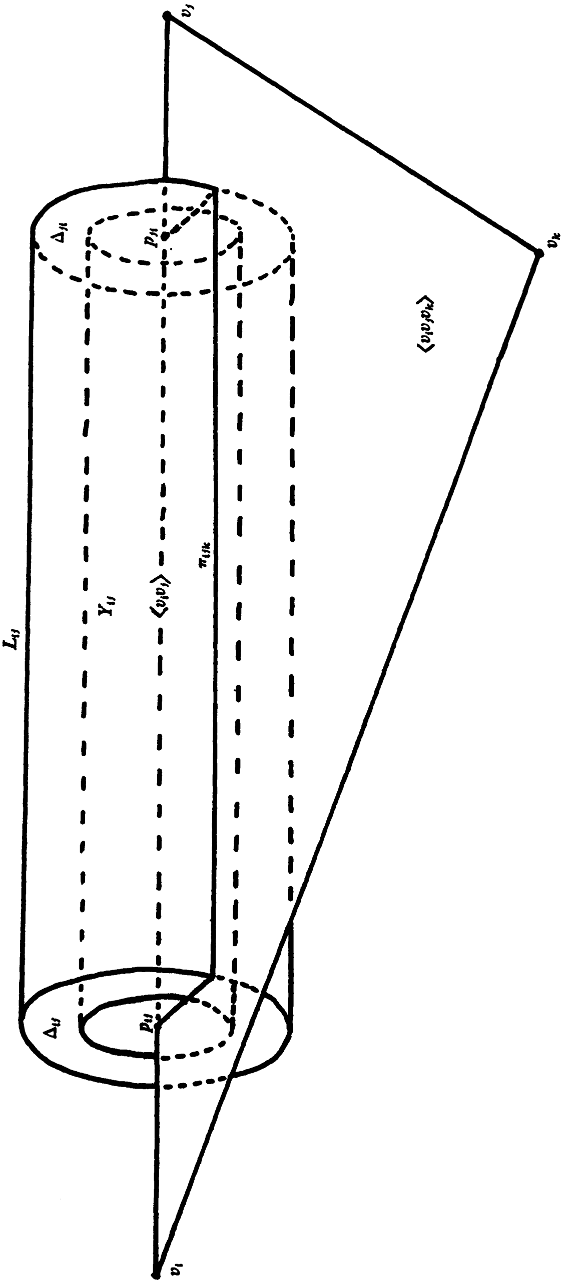
v_j
v_k
$\langle v_i v_j v_k \rangle$
Δ_{ji}
p_{ji}
L_{ij}
Y_{ij}
$\langle v_i v_j \rangle$
π_{ijk}
Δ_{ij}
p_{ij}
v_i

FIGURE 6

If $t=1, 2, \ldots,$ or $\lambda_{ij}-1$, let $\tilde{x}_t\tilde{x}_{t+1}$ and $\tilde{y}_t\tilde{y}_{t+1}$ denote arcs of $\tilde{B}_{ij} \cap \tilde{F}_{ij}$ and $\tilde{B}_{ji} \cap \tilde{F}_{ij}$, respectively, lying in $\tilde{W}_{ijt}$, and let γ^0_{ijt} denote

$$\tilde{x}_t\tilde{x}_{t+1} \cup \mu^0_{ijt} \cup \mu^0_{ij(t+1)} \cup \tilde{y}_t\tilde{y}_{t+1}.$$

Let $\tilde{x}_1\tilde{x}_{\lambda_{ij}}$ and $\tilde{y}_1\tilde{y}_{\lambda_{ij}}$ denote arcs of $\tilde{B}_{ij} \cap \tilde{F}_{ij}$ and $\tilde{B}_{ji} \cap \tilde{F}_{ij}$, respectively, lying in $\tilde{W}_{ij\lambda_{ij}}$, and let $\gamma^0_{ij\lambda_{ij}}$ denote

$$\tilde{x}_1\tilde{x}_{\lambda_{ij}} \cup \mu^0_{ij1} \cup \mu^0_{ij\lambda_{ij}} \cup \tilde{y}_1\tilde{y}_{\lambda_{ij}}.$$

If $t=1, 2, \ldots,$ or λ_{ij}, then $\gamma^0_{ijt} \subset \tilde{W}_{ijt}$; since W_{ijt} is simply connected, then by Lemma 1, so is $\tilde{W}_{ijt}$. Therefore γ^0_{ijt} bounds a polyhedral singular disc Γ_{ijt} contained in $\tilde{W}_{ijt}$.

Let Ω_{ij} denote

$$\tilde{B}_{ij} \cup \tilde{B}_{ji} \cup \left(\bigcup_{t=1}^{\lambda_{ij}} \Gamma_{ij}\right).$$

Then Ω_{ij} is a polyhedral singular annulus such that

(1) Bd $\Omega_{ij}=(\text{Bd } \tilde{\Delta}_{ij}) \cup (\text{Bd } \tilde{\Delta}_{ji})$,

(2) Bd Ω_{ij} has a neighborhood on Ω_{ij} which contains no point of singularity of Ω_{ij}, and

(3) $\Omega_{ij} \subset \tilde{W}_{ij} \cup \tilde{B}_{ij} \cup \tilde{B}_{ji}$.

Let Z_{ij} denote Int $(C_i \cup C_j \cup Y_{ij})$. Then Z_{ij} is a connected open set, v_i and v_j belong to Z_{ij}, and Z_{ij} and $W_{ij} \cup B_{ij} \cup B_{ji}$ are disjoint.

It is clear that $\tilde{Z}_{ij}$ contains a polygonal arc φ_{ij} from $\tilde{v}_i$ to $\tilde{v}_j$ such that φ_{ij} contains exactly one point of each of Int $\tilde{\Delta}_{ij}$ and Int $\tilde{\Delta}_{ji}$.

Suppose now that k is some positive integer such that $\langle v_iv_jv_k\rangle$ is a 2-simplex of T. Let φ_{ijk} denote $\varphi_{ij} \cup \varphi_{ik} \cup \varphi_{jk}$; φ_{ijk} is a polygonal closed curve. Since φ_{ijk} contains exactly one point of each of Int $\tilde{\Delta}_{ij}$ and Int $\tilde{\Delta}_{ji}$, then φ_{ijk} links each of Bd $\tilde{\Delta}_{ij}$ and Bd $\tilde{\Delta}_{ji}$. For a definition of linking of polygonal closed curves as it is used here, and proofs of some elementary properties, see [6, pp. 480–482].

By [14], there exists a nonvoid subset $\mathscr{C}$ of {Bd $\tilde{\Delta}_{ij}$, Bd $\tilde{\Delta}_{ji}$} and a nonsingular disc or annulus A'_{ij} such that

(1) the curves of $\mathscr{C}$ are the boundary curves of A'_{ij}, and

(2) $A'_{ij} \subset \tilde{W}_{ij} \cup \tilde{B}_{ij} \cup \tilde{B}_{ji}$.

Now it will be shown that A'_{ij} is not a disc. Suppose that A'_{ij} is a disc. Then Bd A'_{ij} is either Bd $\tilde{\Delta}_{ij}$ or Bd $\tilde{\Delta}_{ji}$. Suppose Bd $A_{ij}=$ Bd $\tilde{\Delta}_{ij}$. Since φ_{ijk} links Bd $\tilde{\Delta}_{ij}$, it follows that φ_{ijk} intersects A'_{ij}. However, since $A'_{ij} \subset \tilde{W}_{ij} \cup \tilde{B}_{ij} \cup \tilde{B}_{ji}$, $\varphi_{ijk} \subset \tilde{Z}_{ij} \cup \tilde{Z}_{ik} \cup \tilde{Z}_{jk}$, and $\tilde{W}_{ij} \cup \tilde{B}_{ij} \cup \tilde{B}_{ji}$ and $\tilde{Z}_{ij} \cup \tilde{Z}_{ik} \cup \tilde{Z}_{jk}$ are disjoint, this is a contradiction. Similarly, it is impossible that Bd $A'_{ij}=$ Bd $\tilde{\Delta}_{ji}$. Therefore, A'_{ij} is an annulus, and

$$\text{Bd } A'_{ij} = \text{Bd } \tilde{\Delta}_{ij} \cup \text{Bd } \tilde{\Delta}_{ji}.$$

It is clear that

(1) $A'_{ij} \cap \tilde{C}_i=$ Bd $\tilde{\Delta}_{ij}$,

(2) $A'_{ij} \cap \tilde{C}_j=$ Bd $\tilde{\Delta}_{ji}$, and

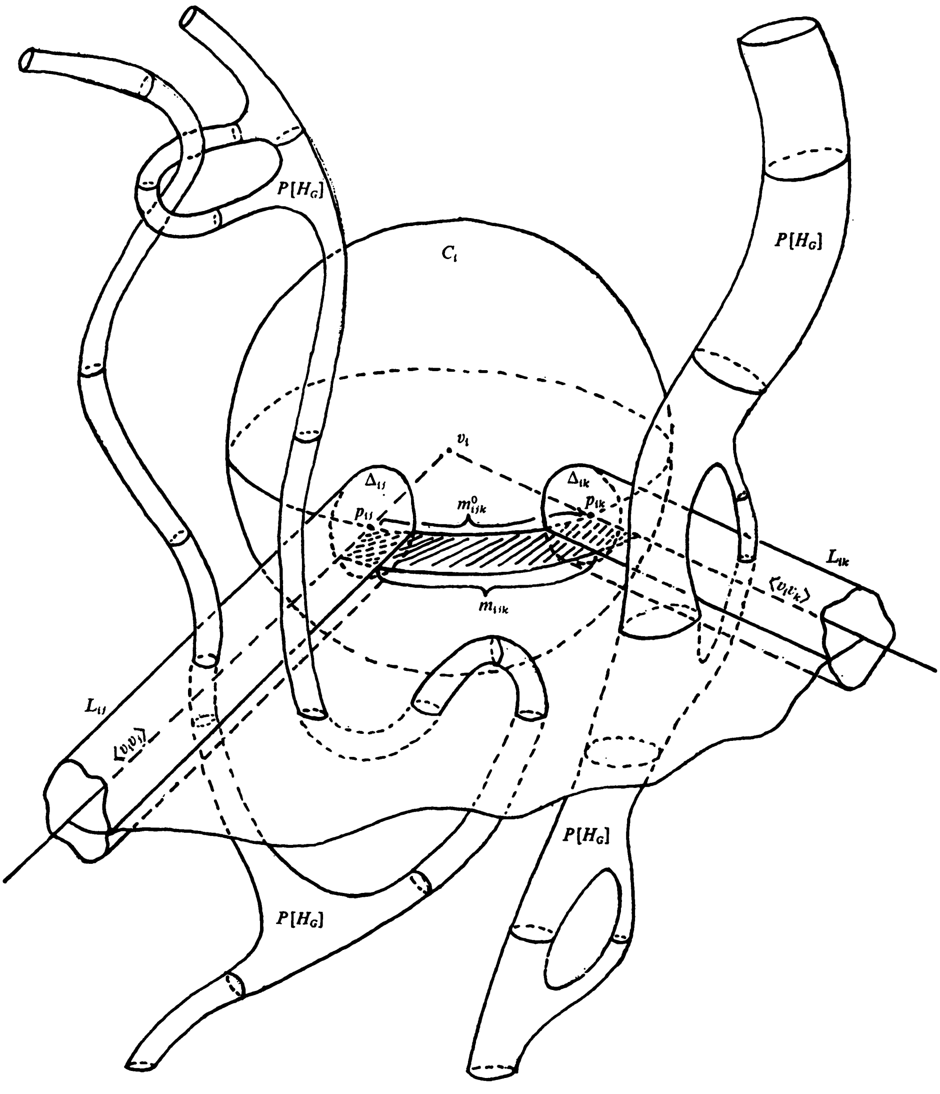

FIGURE 7

(3) if $t = 1, 2, \ldots$ and t is neither i nor j, then $\tilde{C}_t$ and A'_{ij} are disjoint.

Step 6. We may now construct a set K' in M which will be a tubular neighborhood of the carrier Σ_1 of the 1-skeleton of Σ, and which corresponds to the tubular neighborhood K in N.

Suppose that $\langle v_i v_j \rangle$ is a 1-simplex of T. It is clear that $A'_{ij} \cup \tilde{\Delta}_{ij} \cup \tilde{\Delta}_{ji}$ is a polyhedral 2-sphere; let S_{ij} denote $A'_{ij} \cup \tilde{\Delta}_{ij} \cup \tilde{\Delta}_{ji}$. Further if σ is any 3-simplex of T such that $\langle v_i v_j \rangle$ is an edge of σ, then $S_{ij} \subset U_\sigma$. Hence S_{ij} is the boundary of a polyhedral 3-cell L'_{ij} in M such that the following conditions hold:

(1) If $\langle v_s v_t \rangle$ is a 1-simplex of T distinct from $\langle v_i v_j \rangle$ then L'_{ij} and L'_{st} are disjoint.

(2) If $t=1, 2, \ldots$, then $\tilde{C}_t$ and L'_{ij} intersect if and only if $t=i$ or $t=j$, and if $t=i$ or $t=j$, then $L'_{ij} \cap \tilde{C}_t$ is a disc common to Bd L'_{ij} and Bd $\tilde{C}_t$.

Let K' denote

$$\left(\bigcup \{\tilde{C}_i : v_i \textit{ is a vertex of } T\}\right) \cup \left(\bigcup \{L'_{ij} : \langle v_i v_j \rangle \textit{ is a } 1\textit{-simplex of } T\}\right).$$

Step 7. In this step, we prepare for the construction of Σ_1 and Σ_2.

Suppose that $\langle v_i v_j v_k \rangle$ is a 2-simplex of T. Then $\langle v_i v_j v_k \rangle - \text{Int}\,(C_i \cup C_j \cup C_k)$ is a disc ε_{ijk}. There is a polygonal arc α_{ijk} such that

(1) α_{ijk} spans ε_{ijk}, has one endpoint on $\langle v_i v_j \rangle$, the other endpoint on $\langle v_i v_k \rangle$, and is disjoint from the arc $\varepsilon_{ijk} \cap \text{Bd}\, C_i$,

(2) α_{ijk} intersects each of Bd Y_{ij}, Bd Y_{ik}, Bd L_{ij}, and Bd L_{ik} in one point, and α_{ijk} intersects Bd L_{ij} in one point which lies in $B_{ij} - W_{ij}$.

(3) If m^0_{ijk} denotes the closure of the component of $\varepsilon_{ijk} - \alpha_{ijk}$ containing $\varepsilon_{ijk} \cap \text{Bd}\, C_i$, then m^0_{ijk} is disjoint from the closure of $P[H_G]$. See Figure 7.

There are discs m^0_{jik} and m^0_{kij} satisfying analogous conditions and in addition, m^0_{ijk}, m^0_{jik}, and m^0_{kij} are to be mutually disjoint.

Let m_{ijk}, m_{jik}, and m_{kij} denote the discs $m^0_{ijk} - \text{Int}\, K$, $m^0_{jik} - \text{Int}\, K$, and $m^0_{kij} - \text{Int}\, K$, respectively. See Figure 8.

Let τ_{ij} be the arc in Int $\langle v_i v_j v_k \rangle \cap \text{Bd}\, Y_{ij}$ having one endpoint on α_{ijk} and the other on α_{jik}. Let τ_{ik} and τ_{jk} be arcs satisfying analogous conditions. Let α^0_{ijk} be the subarc of α_{ijk} whose endpoints are those of τ_{ij} and τ_{ik} on α_{ijk}. Let θ_{ijk} denote $\tau_{ij} \cup \tau_{ik} \cup \tau_{jk} \cup \alpha^0_{ijk} \cup \alpha^0_{jik} \cup \alpha^0_{kij}$. θ_{ijk} is a polygonal simple closed curve lying in Int $\langle v_i v_j v_k \rangle$. Let δ_{ijk} be the subdisc of $\langle v_i v_j v_k \rangle$ bounded by θ_{ijk}. Observe that

(1) $\delta_{ijk} \subset \text{Int}\, \langle v_i v_j v_k \rangle$ and

(2) δ_{ijk} is disjoint from each of C_i, C_j, and C_k.

Suppose that $\langle v_{i_1} v_{j_1} v_{k_1} \rangle, \langle v_{i_2} v_{j_2} v_{k_2} \rangle, \ldots$ are the distinct 2-simplexes of T. There exist mutually disjoint open sets $V_{i_1 j_1 k_1}, V_{i_2 j_2 k_2}, \ldots$ such that if $\langle v_i v_j v_k \rangle$ is a 2-simplex of T, then

(1) V_{ijk} is obtained by a slight thickening of δ_{ijk}, $\delta_{ijk} \subset V_{ijk}$, and V_{ijk} is simply connected,

(2) if v_t is a vertex of T, C_t and V_{ijk} are disjoint,

(3) $V_{ijk} \cap T_2 \subset \text{Int}\, \langle v_i v_j v_k \rangle$,

(4) $V_{ijk} \cup \text{Int}\, Y_{ij} \cup \text{Int}\, Y_{ik} \cup \text{Int}\, Y_{jk}$ is a simply connected, connected open set,

(5) each of $V_{ijk} \cap \text{Int}\, Y_{ij}$, $V_{ijk} \cap \text{Int}\, Y_{ik}$, and $V_{ijk} \cap \text{Int}\, Y_{jk}$ is connected,

(6) each of $V_{ijk} \cap m_{ijk}$, $V_{ijk} \cap m_{jik}$, and $V_{ijk} \cap m_{kij}$ is connected, and

(7) if σ is any 3-simplex having $\langle v_i v_j v_k \rangle$ as a face, then $V_{ijk} \subset \text{Int}\, X_\sigma$. See Figure 9; this shows a part of V_{ijk} near Bd C_i. Bd C_i is omitted and parts of A_{ij} and A_{ik} are cut away.

Step 8. We shall now construct, for each 1-simplex $\langle v_i v_j \rangle$ of T, arcs π'_{ijk} on A'_{ij} that correspond to the arcs π_{ijk} of A_{ij}.

Suppose that $\langle v_i v_j \rangle$ is a 1-simplex of T and that $\langle v_i v_j v_k \rangle$ is a 2-simplex of T having

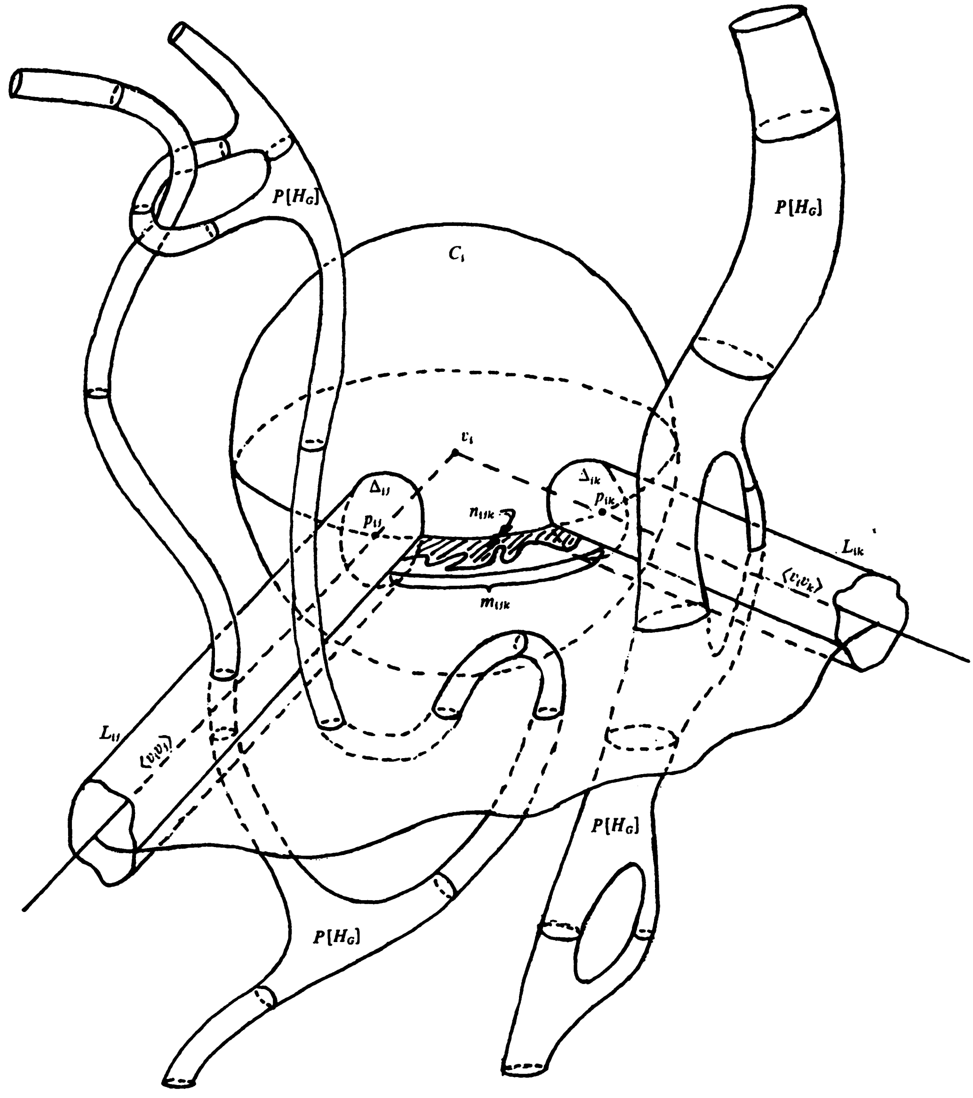

FIGURE 8

$\langle v_i v_j \rangle$ as an edge. Let w_{ij} and w_{ik} be the endpoints of the arc α_{ijk} on $\langle v_i v_j \rangle$ and $\langle v_i v_k \rangle$, respectively. Let w_{ji} and w_{jk} be the endpoints of the arc α_{jik} on $\langle v_j v_i \rangle$ and $\langle v_j v_k \rangle$, respectively. Let w_{ki} and w_{kj} be the endpoints of the arc α_{kij} on $\langle v_k v_i \rangle$ and $\langle v_k v_j \rangle$, respectively.

Let ρ_{ijk} be a simple closed curve in M which is a union of arcs as follows:

(1) An arc in P^{-1} [Int Y_{ij}] from $\tilde{w}_{ij}$ to $\tilde{w}_{ji}$.

(2) An arc in P^{-1} [Int Y_{ik}] from $\tilde{w}_{ik}$ to $\tilde{w}_{ki}$.

(3) An arc in P^{-1} [Int Y_{jk}] from $\tilde{w}_{jk}$ to $\tilde{w}_{kj}$.

(4) The arcs $\tilde{\alpha}_{ijk}$, $\tilde{\alpha}_{jik}$, and $\tilde{\alpha}_{kij}$.

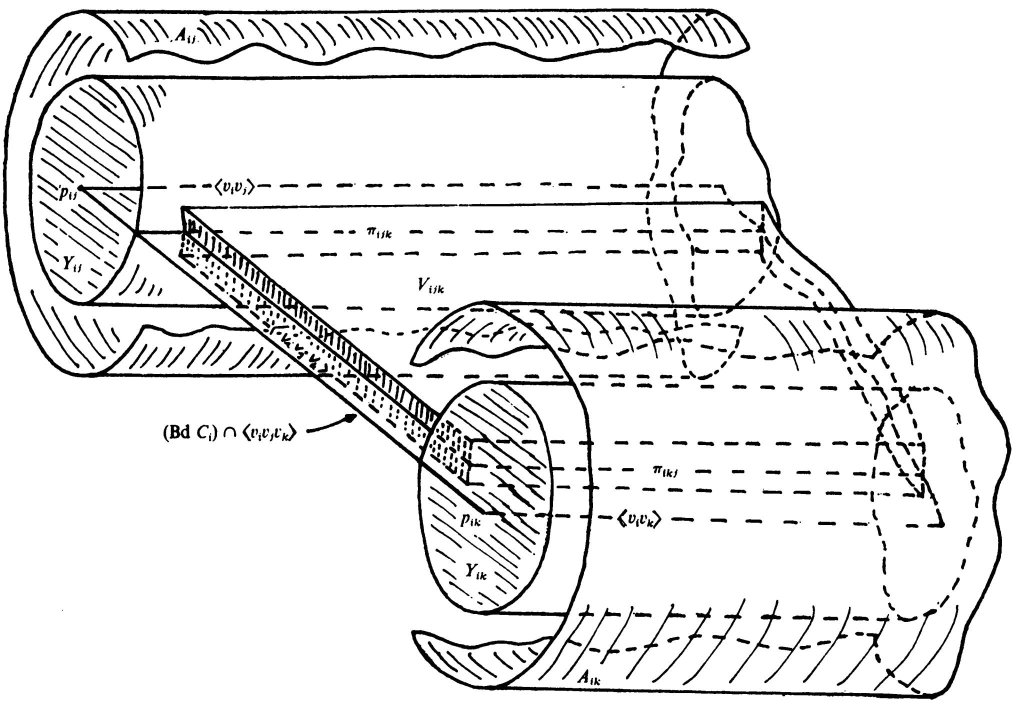

FIGURE 9

The simple closed curve ρ_{ijk} lies in

$$P^{-1}[V_{ijk} \cup \text{Int } Y_{ij} \cup \text{Int } Y_{ik} \cup \text{Int } Y_{jk}].$$

Now consider the arc α_{ijk} and Bd L_{ij}. It follows from the construction of α_{ijk} that there is a point z_{ijk} such that $\alpha_{ijk} \cap \text{Bd } L_{ij} = \{z_{ijk}\}$; further, $z_{ijk} \in \text{Int } B_{ij}$ and $z_{ijk} \notin W_{ij}$. Similarly, there is a point z_{ijk} such that $z_{ijk} \in \text{Int } B_{ik}$, $z_{ikj} \notin W_{ik}$, and $\alpha_{ijk} \cap \text{Bd } L_{ik} = \{z_{ikj}\}$. There are points z_{jik} and z_{jki} of α_{jik}, and z_{kij} and z_{kji} of α_{kij} having analogous properties.

It follows from the construction of the 2-sphere S_{ij} (in Step 6) that ρ_{ijk} has exactly two points, $\tilde{z}_{ijk}$ and $\tilde{z}_{jik}$, in common with S_{ij}.

Let σ be some 3-simplex of T having $\langle v_i v_j v_k \rangle$ as a face. Then different components of $\rho_{ijk} - \{\tilde{z}_{ijk}, \tilde{z}_{jik}\}$ lie in different components of $U_\sigma - S_{ij}$, and hence in different components of $M - S_{ij}$.

Since

$$V_{ijk} \cup (\text{Int } Y_{ij}) \cup (\text{Int } Y_{ik}) \cup (\text{Int } Y_{jk})$$

is simply connected, then by Lemma 1,

$$P^{-1}[V_{ijk} \cup (\text{Int } Y_{ij}) \cup (\text{Int } Y_{ik}) \cup (\text{Int } Y_{jk})]$$

is simply connected. Hence ρ_{ijk} bounds a polyhedral singular disc ρ^*_{ijk} lying in

$$P^{-1}[V_{ijk} \cup (\text{Int } Y_{ij}) \cup (\text{Int } Y_{ik}) \cup (\text{Int } Y_{jk})],$$

in U_σ, and in general position relative to S_{ij}. Note that ρ^*_{ijk} is disjoint from $\tilde{\Delta}_{ij}$ and $\tilde{\Delta}_{ji}$.

By Lemma 4, there is an arc $\tilde{z}_{ijk}\tilde{z}_{jik}$ on S_{ij} from $\tilde{z}_{ijk}$ to $\tilde{z}_{jik}$ and contained in $S_{ij} \cap \rho^*_{ijk}$. Since S_{ij} is disjoint from

$$P^{-1}[(\text{Int } Y_{ij}) \cup (\text{Int } Y_{ik}) \cup (\text{Int } Y_{jk})]$$

by construction, it follows that $\tilde{z}_{ijk}\tilde{z}_{jik}$ lies in $\tilde{V}_{ijk}$. Further, $\tilde{z}_{ijk}\tilde{z}_{jik} \subset \text{Int } A'_{ij}$.

Recall that $A_{ij} \cap \langle v_i v_j v_k \rangle$ is an arc π_{ijk}. Let q_{ijk} and q_{jik} denote the endpoints of π_{ijk} on Bd C_i and Bd C_j, respectively. Clearly z_{ijk} and z_{jik} belong to Int π_{ijk}. It follows that the subarcs $q_{ijk}z_{ijk}$ and $q_{jik}z_{jik}$ of π_{ijk} are disjoint from Cl $P[H_G]$.

Let t_{ijk} be the last point on $\tilde{z}_{ijk}\tilde{z}_{jik}$ (in the order from $\tilde{z}_{ijk}$ to $\tilde{z}_{jik}$ on $\tilde{z}_{ijk}\tilde{z}_{jik}$) of $\tilde{q}_{ijk}\tilde{z}_{ijk}$, and let t_{jik} be the first point on $\tilde{z}_{jik}\tilde{q}_{jik}$ (in the order from $\tilde{z}_{jik}$ to $\tilde{q}_{jik}$ on $\tilde{z}_{jik}\tilde{q}_{jik}$) of the subarc $t_{ijk}\tilde{z}_{jik}$ of $\tilde{z}_{ijk}\tilde{z}_{jik}$. Then let π'_{ijk} denote the union of

(1) the subarc $\tilde{q}_{ijk}t_{ijk}$ of $\tilde{q}_{ijk}\tilde{z}_{ijk}$,

(2) the subarc $t_{ijk}t_{jik}$ of $\tilde{z}_{ijk}\tilde{z}_{jik}$, and

(3) the subarc $t_{jik}\tilde{q}_{jik}$ of $\tilde{z}_{jik}\tilde{q}_{jik}$.

Consider the disc m_{ijk}. Since $V_{ijk} \cap m_{ijk}$ is connected, there is an arc β_{ijk} spanning $\tilde{m}_{ijk}$, having endpoints t_{ijk} and t_{ikj}, and lying in $\tilde{V}_{ijk}$. Let n_{ijk} be the subdisc of m_{ijk} which is the closure of the component of $m_{ijk} - P[\beta_{ijk}]$ containing points on Bd C_i. Then n_{ijk} is disjoint from Cl $P[H_G]$ because m_{ijk} is disjoint from Cl $P[H_G]$.

There exist arcs β_{ijk} and β_{kij}, and discs n_{jik} and n_{kij} with analogous properties.

Let π''_{ijk} denote the subarc $t_{ijk}t_{jik}$ of π'_{ijk}, and let π''_{jki} and π''_{ikj} denote subarcs of π'_{jki} and π'_{ikj}, respectively, defined analogously.

Let μ_{ijk} denote

$$\beta_{ijk} \cup \beta_{jik} \cup \beta_{kij} \cup \pi''_{ijk} \cup \pi''_{jki} \cup \pi''_{ikj}.$$

Then μ_{ijk} is a polygonal simple closed curve lying in $\tilde{V}_{ijk}$.

Step 9. In this step, we shall describe Σ_0 and Σ_1, and begin the construction of Σ_2. Σ_0 is defined to be

$$\{\tilde{v}_i : v_i \text{ is a vertex of } T\}.$$

Suppose that $\langle v_i v_j \rangle$ is a 1-simplex of T. Recall that $\langle v_i v_j v_{k_1} \rangle, \langle v_i v_j v_{k_2} \rangle, \ldots$, and $\langle v_i v_j v_{k_{d(ij)}} \rangle$ are the 2-simplexes of T having $\langle v_i v_j \rangle$ as an edge. If $s = 1, 2, \ldots$, or $d(ij)$, there is an arc π'_{ijk_s} with one endpoint on Bd $\tilde{\Delta}_{ij}$, the other on Bd $\tilde{\Delta}_{ji}$, spanning A'_{ij}, and lying in $\tilde{V}_{ijk_s}$; see Step 8.

There is a piecewise linear homeomorphism g_{ij} from L_{ij} onto L'_{ij} such that (1) $g_{ij} \mid (\Delta_i \cup \Delta_j) = P^{-1} \mid (\Delta_i \cup \Delta_j)$ and (2) if $k = 1, 2, \ldots$, or $d(ij)$, $g_{ij}[\pi_{ijk}] = \pi'_{ijk}$. Now let g be the piecewise linear homeomorphism from K onto K' defined as follows:

(1) If v_i is a vertex of T, $g|C_i = P^{-1}|C_i$.

(2) If $\langle v_i v_j \rangle$ is a 1-simplex of T, $g|L_{ij} = g_{ij}$.

For each 1-simplex $\langle v_i v_j \rangle$ of T, let e_{ij} denote $g[\langle v_i v_j \rangle]$. Let Σ_1 denote

$$\bigcup \{e_{ij} : \langle v_i v_j \rangle \textit{ is a } 1\textit{-simplex of } T\}.$$

Step 10. Suppose that $\langle v_i v_j v_k \rangle$ is a 2-simplex of T. Corresponding to $\langle v_i v_j v_k \rangle$, there is to be a 2-simplex D_{ijk} of Σ. The boundary of D_{ijk} is to be γ_{ijk} where $\gamma_{ijk} = g[\text{Bd } \langle v_i v_j v_k \rangle]$. We get D_{ijk} by constructing two sets as follows: (1) A nonsingular annulus O_{ijk} with boundary curves γ_{ijk} and μ_{ijk} (described in Step 8) and (2) a singular disc R_{ijk} with boundary μ_{ijk} and disjoint from γ_{ijk}. In this step, we describe O_{ijk} and a certain singular disc R^0_{ijk} to be used in the construction of R_{ijk}.

First we shall describe the annulus O_{ijk}. Let γ_{ijk} denote $g[\text{Bd } \langle v_i v_j v_k \rangle]$; $\gamma_{ijk} = e_{ij} \cup e_{ik} \cup e_{jk}$. Let O_{ijk} denote

$$g[K \cap \langle v_i v_j v_k \rangle] \cup n_{ijk} \cup n_{jik} \cup n_{kji};$$

the discs n_{ijk}, n_{jik}, and n_{kij} are described in Step 8. Clearly O_{ijk} is a polyhedral annulus and Bd $O_{ijk} = \gamma_{ijk} \cup \mu_{ijk}$. Further, if $\langle v_s v_t v_u \rangle$ is a 2-simplex of T distinct from $\langle v_i v_j v_k \rangle$, then O_{ijk} and O_{stu} intersect if and only if $\langle v_i v_j v_k \rangle$ and $\langle v_s v_t v_u \rangle$ intersect, and (1) if $\langle v_i v_j v_k \rangle \cap \langle v_s v_t v_u \rangle$ is a 1-simplex $\langle v_l v_r \rangle$ of T, then $O_{ijk} \cap O_{stu} = e_{lr}$, and (2) if, for some vertex v_l of T, $\langle v_i v_j v_k \rangle \cap \langle v_s v_t v_u \rangle = \{v_l\}$, then $O_{ijk} \cap O_{stu} = \{\tilde{v}_l\}$.

We shall now construct the singular disc R_{ijk}. We first construct strips to be attached to A'_{ij}, A'_{ik}, and A'_{jk}; the purpose of these strips is to insure that R_{ijk} is in normal position relative to each of L'_{ij}, L'_{ik}, and L'_{jk}. Let s^0_{ijk} be a disc lying in $\tilde{V}_{ijk}$ whose boundary is the union of the following arcs:

(1) The arc $\mu_{ijk} \cap A'_{ij}$.
(2) A very short subarc $t_{ijk}t'_{ijk}$ of the arc $t_{ijk}t_{ikj}$.
(3) A very short subarc $t_{ijk}t'_{jik}$ of the arc $t_{jik}t_{jki}$.
(4) An arc β^*_{ijk} from t'_{ijk} to t'_{jik}, lying in $\tilde{V}_{ijk}$, disjoint from

$$\tilde{C}_i \cup \tilde{C}_j \cup \tilde{C}_k \cup L'_{ij} \cup L'_{ik} \cup L'_{jk},$$

and "parallel to" the arc $\mu_{ijk} \cap A'_{ij}$. We think of s^0_{ijk} as a narrow strip attached to A'_{ij}.

There exist strips s^0_{jik} and s^0_{kij}, points t'_{jki} of the arc $t_{jik}t_{jki}$, t'_{kij} and t'_{kji} of the arc $t_{kij}t_{kji}$, and t'_{ikj} of the arc $t_{ijk}t_{ikj}$, and arcs β^*_{jki} from t'_{jki} to t'_{kji}, and β^*_{kij} from t_{kij} to t_{kji}, having analogous properties.

Let μ'_{ijk} denote the simple closed curve

$$\beta^*_{ijk} \cup \beta^*_{jik} \cup \beta^*_{kij} \cup t'_{ijk}t'_{ikj} \cup t'_{jik}t'_{jki} \cup t'_{kij}t'_{kji};$$

μ'_{ijk} lies in $\tilde{V}_{ijk}$. Since $\tilde{V}_{ijk}$ is simply connected, there is a polyhedral singular disc R'_{ijk} such that Bd $R'_{ijk} = \mu'_{ijk}$ and $R'_{ijk} \subset \tilde{V}_{ijk}$. Let R^0_{ijk} denote

$$R'_{ijk} \cup s^0_{ijk} \cup s^0_{jik} \cup s^0_{kij};$$

R^0_{ijk} is a polyhedral singular disc lying in $\tilde{V}_{ijk}$ and such that (1) Bd $R^0_{ijk}=\mu_{ijk}$ and (2) R^0_{ijk} is in normal position relative to each of L'_{ij}, L'_{ik} and L'_{jk}.

Note that if $\langle v_iv_jv_k\rangle$ and $\langle v_sv_tv_u\rangle$ are distinct 2-simplexes of T, then R^0_{ijk} and R^0_{stu} are disjoint.

Step 11. It may not be true that for each 1-simplex $\langle v_iv_jv_k\rangle$ of T, R^0_{ijk} and γ_{ijk} are disjoint. In this step, we use Lemma 5 to obtain mutually disjoint singular discs R_{ijk}, each disjoint from Σ_1.

Consider the 1-simplexes $\langle v_{i_1}v_{j_1}\rangle$, $\langle v_{i_2}v_{j_2}\rangle,\dots$ of T. Let U'_1, $U'_2,\dots$ be mutually disjoint open sets in M such that if $s=1, 2,\dots$, then

(1) Int $A'_{i_sj_s}\subset U'_s$,

(2) for each vertex v_t of T, U'_s is disjoint from $\tilde{C}_t$,

(3) if $\langle v_iv_jv_k\rangle$ is a 2-simplex of T, then R^0_{ijk} intersects U'_s if and only if R^0_{ijk} intersects $A'_{i_sj_s}$, and

(4) if σ is any 3-simplex of T having $\langle v_{i_s}v_{j_s}\rangle$ as an edge, then $U'_s\subset U_\sigma$.

Recall that if $\langle v_{i_t}v_{j_t}\rangle$ is a 1-simplex of T, there are $d(i_tj_t)$ distinct 2-simplexes of T having $\langle v_{i_t}v_{j_t}\rangle$ as an edge. If $s=1, 2,\dots$, or $d(i_tj_t)$ and $\langle v_{i_t}v_{j_t}v_{k_s}\rangle$ is a 2-simplex of T, let R^0_{ts} denote $R^0_{i_tj_tk_s}$.

First consider $\langle v_{i_1}v_{j_1}\rangle$. By Lemma 5, there exist mutually disjoint polyhedral singular discs R^1_{11}, $R^1_{12},\dots$, and $R^1_{1d(i_1j_1)}$ such that if $s=1, 2,\dots$, or $d(i_1j_1)$, then

(1) $R^1_{1s}\cap L'_{i_1j_1}=\pi'_{i_1j_1k_{1s}}$,

(2) Bd $R^1_{1s}=$ Bd R^0_{1s}, and

(3) $R^1_{1s}\subset U'_1\cup R^0_{1s}$.

If $t=2, 3,\dots$, then for each 2-simplex $\langle v_{i_t}v_{j_t}v_{k_s}\rangle$ of T, let R^1_{ts} denote R^0_{ts}. Then the following statements hold:

(1) If $\langle v_{i_r}v_{j_r}v_{k_s}\rangle$ and $\langle v_{i_t}v_{j_t}v_{k_u}\rangle$ are distinct 2-simplexes of T, then R^1_{rs} and R^1_{tu} are disjoint.

(2) If $\langle v_{i_r}v_{j_r}v_{k_s}\rangle$ is a 2-simplex of T and v_t is a vertex of T, then R^1_{rs} and $\tilde{C}_t$ are disjoint.

(3) If $\langle v_{i_r}v_{j_r}\rangle$ and $\langle v_tv_u\rangle$ are 1-simplexes of T distinct from $\langle v_{i_1}v_{j_1}\rangle$ and s is a positive integer such that $\langle v_{i_r}v_{j_r}v_{k_s}\rangle$ is a 2-simplex of T, then $R^1_{rs}\cap S_{tu}=R^0_{rs}\cap S_{tu}$.

Now consider $\langle v_{i_2}v_{j_2}\rangle$. The process described above, which used $L'_{i_1j_1}$ and the singular discs R^0_{11}, $R^0_{12},\dots$, and $R^0_{1d(i_1j_1)}$, may be repeated using $L'_{i_2j_2}$ and the singular discs R^1_{21}, $R^1_{22},\dots$, and $R^1_{2d(i_2j_2)}$. There result mutually disjoint polyhedral singular discs R^2_{21}, $R^2_{22},\dots$, and $R^2_{2d(i_2j_2)}$. If $t=1, 3, 4,\dots$, then for each 2-simplex $\langle v_{i_t}v_{j_t}v_{k_s}\rangle$ of T, let R^2_{ts} denote R^1_{ts}. The singular discs R^2_{ts} have properties analogous to those of the discs R^1_{ts} mentioned above, and, in addition, the following: For any 2-simplex $\langle v_{i_r}v_{j_r}v_{k_s}\rangle$ of T, if $u=1$ or 2 and $S_{i_uj_u}$ and R^2_{rs} intersect, then $i_u=i_r$, $j_u=j_r$, and $S_{i_uj_u}\cap R^2_{rs}=\pi'_{i_rj_rk_s}$.

This process may be continued, considering $\langle v_{i_3}v_{j_3}\rangle$, $\langle v_{i_4}v_{j_4}\rangle,\dots$. There results, for each 2-simplex $\langle v_iv_jv_k\rangle$ of T, a polyhedral singular disc R_{ijk} such that the following statements hold:

(1) For any 2-simplex $\langle v_i v_j v_k \rangle$ of T, Bd $R_{ijk}=$ Bd R^0_{ijk}, and for each vertex v_t of T, R_{ijk} and $\tilde{C}_t$ are disjoint.

(2) If $\langle v_i v_j v_k \rangle$ and $\langle v_s v_t v_u \rangle$ are distinct 2-simplexes of T, then R_{ijk} and R_{stu} are disjoint.

(3) If $\langle v_i v_j \rangle$ is any 1-simplex of T and $\langle v_s v_t v_u \rangle$ is any 2-simplex of T, then L'_{ij} and R_{stu} intersect if and only if i and j belong to $\{s, t, u\}$ and if R_{stu} intersects L'_{ij}, $R_{stu} \cap L'_{ij}$ is the arc π'_{stu}.

Now we may construct the carrier Σ_2 of the 2-skeleton of Σ. Suppose that $\langle v_i v_j v_k \rangle$ is a 2-simplex of T. It is clear that R_{ijk} does not intersect

$$\text{Int}\,[\tilde{C}_i \cup \tilde{C}_j \cup \tilde{C}_k \cup L'_{ij} \cup L'_{ik} \cup L'_{jk}].$$

Hence $R_{ijk} \cup O_{ijk}$ is a polyhedral singular disc with boundary γ_{ijk}, and there is a neighborhood of γ_{ijk} containing no point of singularity of $R_{ijk} \cup O_{ijk}$. Further, if σ is any 3-simplex of T having $\langle v_i v_j v_k \rangle$ as a face, then $R_{ijk} \cup O_{ijk} \subset U_\sigma$.

With the aid of Dehn's lemma as proved by Papakyriakopoulos [11], it may be shown that for each 2-simplex $\langle v_i v_j v_k \rangle$ of T, there is a polyhedral disc D_{ijk} such that

(1) Bd $D_{ijk}=\gamma_{ijk}$,

(2) there is an annulus on $D_{ijk} \cap O_{ijk}$ having γ_{ijk} as one boundary component,

(3) if $\langle v_s v_t v_u \rangle$ is any 2-simplex of T distinct from $\langle v_i v_j v_k \rangle$, then $D_{ijk} \cap D_{stu} = O_{ijk} \cap O_{stu}$, and

(4) if σ is any 3-simplex of T having $\langle v_i v_j v_k \rangle$ as a face, then $D_{ijk} \subset U_\sigma$.

Define Σ_2 to be

$$\bigcup \{D_{ijk} : \langle v_i v_j v_k \rangle \textit{ is a 2-simplex of } T\}.$$

Step 12. We may now define Σ and construct an embedding h of T_2 into M. Define Σ to be

$$\{\tilde{v}_i : v_i \textit{ is a vertex of } T\} \cup \{e_{ij} : \langle v_i v_j \rangle \textit{ is a 1-simplex of } T\} \cup \{D_{ijk} : \langle v_i v_j v_k \rangle \textit{ is a 2-simplex of } T\}.$$

Let φ be the function from the 2-skeleton of T onto Σ defined as follows: (1) If v_i is a vertex of T, $\varphi(v_i)=\tilde{v}_i$. (2) If $\langle v_i v_j \rangle$ is a 1-simplex of T, $\varphi(\langle v_i v_j \rangle)=e_{ij}$. (3) If $\langle v_i v_j v_k \rangle$ is a 2-simplex of T, $\varphi(\langle v_i v_j v_k \rangle)=D_{ijk}$. Clearly φ is an isomorphism.

Σ is a locally finite collection of subsets of M. This is true because (1) $\{U_\sigma : \sigma \textit{ is a 3-simplex of } T\}$ is locally finite and (2) if σ is any 3-simplex of T and τ is any proper face of σ, $\varphi(\tau) \subset U_\sigma$.

It follows that there is a piecewise linear homeomorphism h from T_2 onto Σ_2 such that (1) for each vertex v_i of T, $h(v_i)=\tilde{v}_i$, (2) for each 1-simplex $\langle v_i v_j \rangle$ of T, $h[\langle v_i v_j \rangle]=e_{ij}$, and (3) for each 2-simplex $\langle v_i v_j v_k \rangle$ of T, $h[\langle v_i v_j v_k \rangle]=D_{ijk}$. It is clear that if σ is any 3-simplex of T, $h[\text{Bd } \sigma] \subset U_\sigma$. Recall that the set X_σ is constructed in Step 1 so that if τ is a 3-simplex of T distinct from σ, X_σ contains no vertex of τ. It follows that if τ is any 3-simplex of T distinct from σ, $h[\text{Bd } \tau]$ does not lie in U_σ.

In addition, $U_\sigma \neq U_\tau$. Then by Lemma 8, M and N are homeomorphic. This completes the proof of the theorem.

7. Corollaries of the main result.

COROLLARY 1. *Suppose M is a* 3*-manifold, G is a cellular decomposition of M such that M/G is a* 3*-manifold N, and N has a triangulation T such that* (1) *no vertex of T belongs to* $\mathrm{Cl}\, P[H_G]$ *and* (2) *if σ is any* 3*-simplex of T, there is an open* 3*-cell in M containing $P^{-1}[\sigma]$. Then M and N are homeomorphic.*

Proof. Apply the theorem to each component of M.

COROLLARY 2. *Suppose that M is a* 3*-manifold, G is a cellular decomposition of M such that M/G is a* 3*-manifold N, and $P[H_G]$ is a nowhere dense subset of N. Then M and N are homeomorphic.*

Proof. Since G is a cellular decomposition of M, there is a triangulation T^0 of N such that if σ is any 3-simplex of T^0, $P^{-1}[\sigma]$ lies in some open 3-cell in M. Since $P[H_G]$ is nowhere dense in N, so is $\mathrm{Cl}\, P[H_G]$. A slight adjustment of the vertices of T^0 yields a triangulation T of N such that (1) no vertex of T belongs to $\mathrm{Cl}\, P[H_G]$ and (2) if σ is any 3-simplex of T, $P^{-1}[\sigma]$ lies in an open 3-cell in M. By Corollary 1, M and N are homeomorphic.

Corollary 2 is a generalization of the result announced in [3].

COROLLARY 3. *If M is a* 3*-manifold, G is a cellular decomposition of M such that M/G is a* 3*-manifold N, and $P[H_G]$ lies in a closed set of dimension at most two, then M and N are homeomorphic.*

The following corollary is a generalization of Theorem 3 of [2].

COROLLARY 4. *If M is a* 3*-manifold, G is a cellular decomposition of M, M/G is a* 3*-manifold N, and $P[H_G]$ is contained in a closed* 0*-dimensional set, then M and N are homeomorphic.*

The next two corollaries deal with point-like decompositions of E^3 and S^3. It is clear that if M is either E^3 or S^3, G is a point-like decomposition of M, and σ is a compact proper subset of M, then $P^{-1}[\sigma]$ lies in an open 3-cell in M. It was mentioned in §2 that in E^3 and S^3, "point-like" and "cellular" are equivalent.

COROLLARY 5. *Suppose that M is either E^3 or S^3, G is a point-like decomposition of M, and M/G is a* 3*-manifold N such that N has a triangulation, no vertex of which belongs to* $\mathrm{Cl}\, P[H_G]$*. Then M and N are homeomorphic.*

COROLLARY 6. *Suppose that G is a point-like decomposition of S^3, S^3/G is a* 3*-manifold N, and there is an open set U such that U and $P[H_G]$ are disjoint. Then N is homeomorphic to S^3.*

Proof. N is a compact connected 3-manifold. Hence N has a triangulation, each vertex of which lies in U. Then Corollary 6 follows from Corollary 5.

8. **Cellular maps of compact 3-manifolds.** In this section, we shall formulate some of the results of this paper in terms of continuous functions rather than upper semicontinuous decompositions. We restrict our attention to compact manifolds.

If M is a 3-manifold, by a *cellular* map from M into a space is meant a continuous function with domain M such that if $y \in f[M]$, $f^{-1}[y]$ is a cellular subset of M. If f is a continuous function from a space X onto a space Y, then a point p of Y is a *nonsingular point* of f if and only if there is an open neighborhood U of p such that $f^{-1}|U$ is a homeomorphism. A point p of Y is a *singular point* of f if and only if p is not a nonsingular point of f. The set of all singular points of f is closed in Y.

COROLLARY 7. *Suppose that f is a cellular map from a compact 3-manifold M onto a 3-manifold N. Suppose N has a triangulation T such that* (1) *no vertex of T is a singular point of f and* (2) *if σ is any 3-simplex of T, then $f^{-1}[\sigma]$ lies in an open 3-cell in M. Then M and N are homeomorphic.*

COROLLARY 8. *Suppose that f is a cellular map from a compact 3-manifold M onto a 3-manifold N such that the set of all singular points of f is nowhere dense in N. Then M and N are homeomorphic.*

COROLLARY 9. *If f is a cellular map from S^3 onto a 3-manifold N and the set of all singular points of f is a proper subset of N, then N is homeomorphic to S^3.*

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